# MAC-CPTM Situations Project

## Situation 01: Sine 32°

Prepared at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
29 June 2005 – Pat Wilson, Heather Godine, and Jeanne Shimizu

Edited at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
12 February 2007 – Evan McClintock
23 February 2007 – Rose Mary Zbiek, M. Kathleen Heid

## Prompt

After completing a discussion on special right triangles (30°-60°-90° and 45°-45°-90°), the teacher showed students how to calculate the sine of various angles using the calculator.

A student then asked, "How could I calculate sin (32°) if I do not have a calculator?"

## Commentary

The set of foci provide interpretations of sine as a ratio and sine as a function. The first two foci highlight $\sin(\theta)$ as a ratio, appealing to right-triangle and unit-circle trigonometry. The second two foci highlight $\sin(x)$as a function and use tangent and secant lines to approximate $\sin(x)$.

## Mathematical Foci

**Mathematical Focus 1**

*Ratios of lengths of sides of right triangles can be used to compute and approximate trigonometric function values.*

A ratio of measures of legs of a right triangle with an acute angle of measure x° can be used to *approximate* sin($x$). Sin($x$) can be approximated by sketching a 32°-58°-90° right triangle with a protractor or with software such as Geometer's Sketchpad, measuring the length of the hypotenuse and leg opposite the 32° angle, and computing the sine ratio.

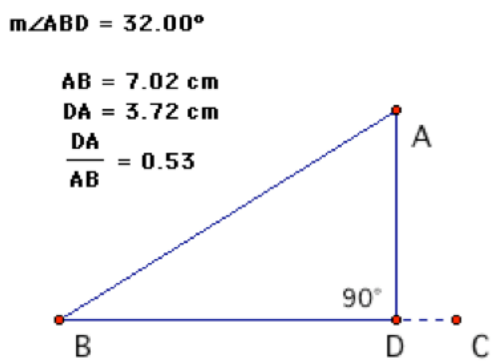


Figure 1

Hence, $\sin(32°) \approx 0.53$.

**Mathematical Focus 2**

*Coordinates of points on the unit circle represent ordered pairs of the form $(\cos(\theta), \sin(\theta))$ that can be used to approximate trigonometric values.*

The unit circle is the locus of all points one unit from the origin $(0,0)$. The equation for a circle with radius 1 centered at the origin is $x^2 + y^2 = 1$. Consider the angle $\theta$ in standard position formed by the x-axis and a ray from the origin to a point A on the unit circle. Then, $\cos(\theta) = \frac{x}{1}$ and $\sin(\theta) = \frac{y}{1}$. Hence, the coordinates of A are $(\cos(\theta), \sin(\theta))$, and another equation for a circle with radius 1 centered at the origin is $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$.

Therefore, the signed length measure of segment AD is equal to sin (32°).

The signed length measure of segment AD is approximately 0.53 and so, $\sin(32°) \approx 0.53$

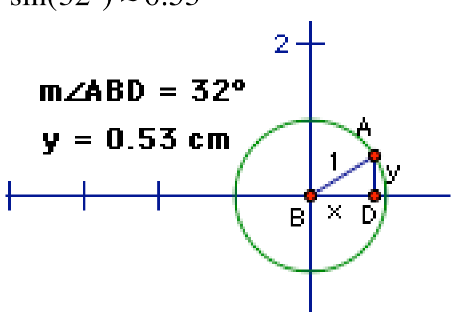


Figure 2

**Mathematical Focus 3**

*A continuous function, such as $f(x) = \sin(x)$, can be represented locally by a linear function and that linear function can be used to approximate local values of the original function.*

The function $f(x) = \sin(x)$ is not a linear function; however, linear functions can be used to approximate non-linear functions over sufficiently small intervals.
Measuring angles in radians:
$180°$ is equivalent to $\pi$ radians, Therefore:
$30°$ is equivalent to $\frac{30\pi}{180} = \frac{\pi}{6}$, or 0.5236 radians
$32°$ is equivalent to $\frac{32\pi}{180} = \frac{8\pi}{45}$, or 0.5585 radians
$45°$ is equivalent to $\frac{45\pi}{180} = \frac{\pi}{4}$, or 0.7854 radians
Figure 3 shows the graph of the function $f(x) = \sin(x)$ and the graph of the secant line $\overleftrightarrow{AB}$, where the coordinates of A are $\left(\frac{\pi}{6}, \sin\left(\frac{\pi}{6}\right)\right) = \left(\frac{\pi}{6}, \frac{1}{2}\right) = \left(\frac{\pi}{6}, 0.5\right)$ and the coordinates of B are $\left(\frac{\pi}{4}, \sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) \approx \left(\frac{\pi}{4}, 0.7071\right)$. Since the function $f(x) = \sin(x)$ is approximately linear between points A and B, the values of the secant line $\overleftrightarrow{AB}$ provide reasonable approximations for the values of $f(x) = \sin(x)$ between points A and B. Since sin(x) is concave down in the interval for x of $\left\langle \frac{\pi}{6}, \frac{\pi}{4} \right\rangle$, the estimate for sin(32°) will be an underestimate.

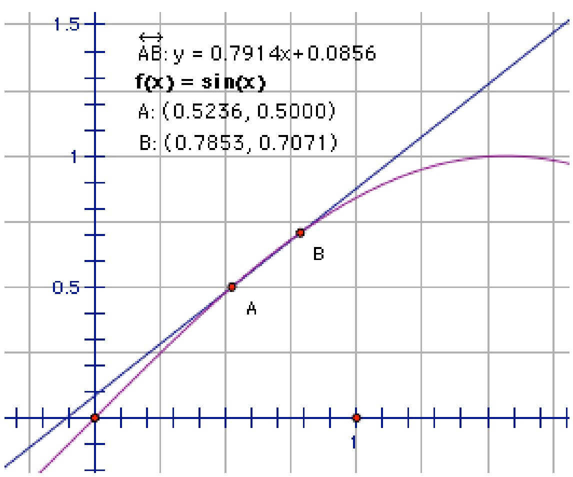

Figure 3

In Figure 4 point D on secant line $\overleftrightarrow{AB}$ with coordinates $(0.5585, 0.5276)$ provides a reasonable approximation for the location of point C on $f(x) = \sin(x)$ with coordinates $(0.5585, \sin(0.5585))$.

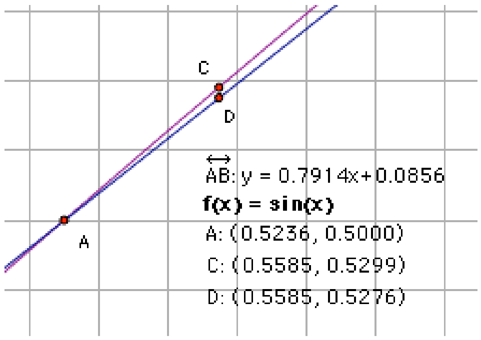

Figure 4

Therefore, $\sin(32°) \approx 0.5276$.

An approximation for $\sin(32°)$ can also be found by using the equation for secant line $\overleftrightarrow{AB}$. Since secant line $\overleftrightarrow{AB}$ passes through the points $\left(\frac{\pi}{6}, \sin\left(\frac{\pi}{6}\right)\right)$ and $\left(\frac{\pi}{4}, \sin\left(\frac{\pi}{4}\right)\right)$, its equation can be determined as follows:

$$y - 0.5 = \frac{0.7071 - 0.5}{0.7853 - 0.5236}(x - 0.5236)$$
$$y = 0.7914(x - 0.5236) + 0.5$$

Therefore, $\sin(32°) \approx y(0.5585) = 0.5276$.

**Mathematical Focus 4**

*Given a differentiable function and a line tangent to the function at a point, values of the tangent line will approximate values of the function near the point of tangency.*

Since the function $f(x) = \sin(x)$ is differentiable, given a point $(a, \sin(a))$ on $f(x) = \sin(x)$, the line tangent to $f(x) = \sin(x)$ at $(a, \sin(a))$ can be used to approximate $(a, \sin(a))$ at a nearby point with x-coordinate $a + dx$. When $dx$ is small, the value of $\sin(a + dx)$ and the value of the tangent line at the point with x-coordinate $a + dx$ will be very close. Using radian measure, 32° is equivalent to $\frac{32\pi}{180} = \frac{8\pi}{45}$, or 0.5585 radians

Consider a geometric interpretation of differentials $dx$ and $dy$ and their relation to $\Delta x$ and $\Delta y$ where a tangent line is used to approximate $f(x)$ near a given value.

$$f'(x) \approx \frac{\Delta y}{\Delta x} \rightarrow \Delta y \approx (\Delta x) f'(x)$$

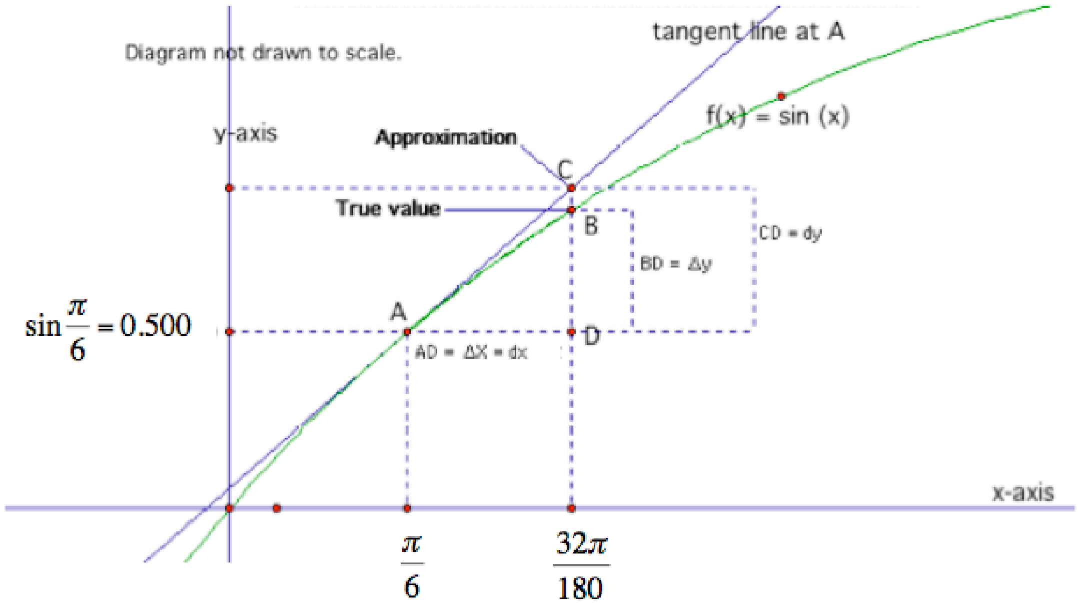


Figure 5

Since $(a, f(a)) = \left(\frac{\pi}{6}, \sin\left(\frac{\pi}{6}\right)\right)$ and $f'(x) = \cos(x)$,

Then,

$\Delta y \approx (\Delta x) f'(x) \Rightarrow$

$$\Rightarrow \sin\left(\frac{32\pi}{180}\right) - \sin\left(\frac{\pi}{6}\right) = \left(\frac{32\pi}{180} - \frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \sin\left(\frac{32\pi}{180}\right) - \sin\left(\frac{\pi}{6}\right) \approx 0.0302$$

$$\Rightarrow \sin\left(\frac{32\pi}{180}\right) \approx 0.0302 + \sin\left(\frac{\pi}{6}\right) = 0.5302$$

## Post Commentary

Although they differ in the use of ratios versus the use of lines as approximation tools, all four methods involve approximations. The ratio methods depend on a definition of the trigonometric functions and therefore are not generalizeable to other types of functions while the line methods depend on characteristics of continuous functions and therefore can be used in many other circumstances. The methods also differ in terms of how they depend on additive and multiplicative structures.